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L^γ Inequalities for the Polar Derivative of Polynomials

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Abstract

In this paper, firstly, we obtain an inequality in L^γ analogue concerning the polar derivative for a polynomial $p(\xi) = \sum_{\nu=0}^{m} c_\nu \xi^\nu$ of degree m having no zero in $|\xi| < r$, $r \geq 1$ proved by Govil et al. [15]. Secondly, we also prove L^γ version for the polar derivative of an ordinary inequality for a polynomial having all its zeros in $|\xi| \leq r$, $r \leq 1$ proved in that same paper. Our results generalize and improve some known inequalities.

Keywords: polynomials; polar derivative; L^γ inequality; Bernstein's inequality.

1 Introduction

We consider P_m the class of polynomials $p(\xi)$ of degree m. For $p \in P_m$, we denote

$$\|p\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \qquad \gamma > 0. \tag{1}$$

According to a fact of analysis [25, 26], we have

$$\lim_{\gamma \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} = \max_{|\xi|=1} |p(\xi)|. \tag{2}$$

Hence, we can duly denote

$$\|p\|_\infty = \max_{|\xi|=1} |p(\xi)|. \tag{3}$$

In addition, if we denote $\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|p(e^{i\theta})|d\theta \right\}$, then a simple calculation shows that,

$$\lim_{\gamma \to 0^+} \|p\|_\gamma = \|p\|_0.$$

If $p \in P_m$, Bernstein [6] proved that,

$$\|p'\|_\infty \le m\|p\|_\infty. \tag{4}$$

Equality holds in inequality (4) for $p(\xi) = \alpha\xi^m$, $\alpha \neq 0$. By letting $\gamma \to \infty$, inequality (4) can be obtained from the inequality,

$$\|p'\|_\gamma \le m\|p\|_\gamma, \qquad \gamma > 0. \tag{5}$$

Zygmund [28] obtained inequality (5) for $\gamma \ge 1$ while Arestov [1] proved for $0 < \gamma < 1$.

If $p \in P_m$ has no zero in $|\xi| < 1$, then (4) and (5) can be respectively replaced by

$$\|p'\|_\infty \le \frac{m}{2}\|p\|_\infty, \tag{6}$$

and

$$\|p'\|_\gamma \le \frac{m}{\|1+\xi\|_\gamma}\|p\|_\gamma, \qquad \gamma > 0. \tag{7}$$

Erdöx first conjectured inequality (6) and later verified by Lax [18], whereas de Brujin [7] proved inequality (7) for $\gamma \ge 1$ while Rahman and Schmeisser [23] proved that (7) stays valid for $0 < \gamma < 1$.

For the case, if $p \in P_m$ has all its zeros in $|\xi| \le 1$, Turán [27] showed

$$\|p'\|_\infty \ge \frac{m}{2}\|p\|_\infty. \tag{8}$$

Equality holds for inequalities (6) and (8) for $p(\xi) = \alpha + \beta\xi^m$, such that $|\alpha| = |\beta|$.

Malik [19] generalized (6) by obtaining that if $p \in P_m$ has no zero in $|\xi| < r$, $r \ge 1$ then,

$$\|p'\|_\infty \le \frac{m}{1+r}\|p\|_\infty. \tag{9}$$

Equality holds in inequality (9) for $p(\xi) = (\xi + r)^m$.

Govil and Rahman [14] extended inequality (9) into L^γ analogue by obtaining that

$$\|p'\|_\gamma \leq \frac{m}{\|r+\xi\|_\gamma}\|p\|_\gamma, \qquad \gamma \geq 1. \tag{10}$$

Similar extension was made by Gardner and Weems [11] and Rather [24] independently for $0 < \gamma < 1$ and proved that,

$$\|p'\|_\gamma \leq \frac{m}{\|r+\xi\|_\gamma}\|p\|_\gamma, \qquad 0 < \gamma < 1. \tag{11}$$

Under the same hypothesis of the polynomial, Govil et al. [15] improved (9) by proving that,

$$\|p'\|_\infty \leq m\left\{\frac{m|c_0| + r^2|c_1|}{m|c_0|(1+r^2) + 2r^2|c_1|}\right\}\|p\|_\infty. \tag{12}$$

Aziz and Rather [4] provides the extension of inequality (12) to L^γ version by proving that for every $\gamma > 0$,

$$\|p'\|_\gamma \leq \frac{m}{\|\delta_{r,1}+\xi\|_\gamma}\|p\|_\gamma, \tag{13}$$

where,

$$\delta_{r,1} = \frac{m|c_0|r^2 + |c_1|r^2}{m|c_0| + r^2|c_1|}. \tag{14}$$

Malik [19] obtained the generalization of (8) by proving that if $p \in P_m$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, then,

$$\|p'\|_\infty \geq \frac{m}{1+r}\|p\|_\infty. \tag{15}$$

Malik [20] generalized inequality (15), where the inequality involves integral mean of $|p(\xi)|$ and $|p'(\xi)|$ on $|\xi| = 1$. In fact, he showed that if $p \in P_m$ has all its zeros in $|\xi| \leq 1$, then for every $\gamma > 0$,

$$\|p'\|_\gamma \geq \frac{m}{\|1+\xi\|_\gamma}\|p\|_\gamma. \tag{16}$$

Aziz and Rather [4] further generalized (16) by proving that,

$$m\left\|\frac{p}{p'}\right\|_\gamma \leq \|1 + t_{r,1}\xi\|_\gamma, \tag{17}$$

where,

$$t_{r,1} = \frac{m|c_m|r^2 + |c_{m-1}|}{m|c_m| + |c_{m-1}|}. \tag{18}$$

By involving certain coefficients of the polynomial, Govil et al. [15] obtained the following two results, which the first generalizes (9) and (12) while the second (15).

Theorem 1.1. *If $p \in P_m$ non-vanishing in $|\xi| < r$, $r \geq 1$, then,*

$$\|p'\|_\infty \leq \frac{m}{1+r}\frac{(1-|\lambda|)(1+r^2|\lambda|) + r(m-1)|l-\lambda^2|}{(1-|\lambda|)(1-r+r^2+r|\lambda|) + r(m-1)|l-\lambda^2|}\|p\|_\infty, \tag{19}$$

where,

$$\lambda = \frac{r}{m}\frac{c_1}{c_0}, \qquad \textit{and} \qquad l = \frac{2r^2}{m(m-1)}\frac{c_2}{c_0}.$$

Theorem 1.2. *If $p \in P_m$ having all its zeros in $|\xi| \le r$, $r \le 1$, then,*

$$\|p'\|_\infty \ge \frac{m}{1+r}\frac{(1-|\omega|)(1+r^2|\omega|)+r(m-1)|\Omega-\omega^2|}{(1-|\omega|)(1-r+r^2+r|\omega|)+r(m-1)|\Omega-\omega^2|}\|p\|_\infty, \tag{20}$$

where,

$$\omega = \frac{1}{mr}\frac{\bar{c}_{m-1}}{\bar{c}_m}, \quad \textit{and} \quad \Omega = \frac{2}{m(m-1)r^2}\frac{\bar{c}_{m-2}}{\bar{c}_m}. \tag{21}$$

Now, we define the polar derivative of the polynomial $p(\xi)$ of degree m with respect to α such that α is a complex number or real number, then,

$$D_\alpha p(\xi) = mp(\xi) + (\alpha - \xi)p'(\xi).$$

This polynomial $D_\alpha p(\xi)$ will be of degree at most $m-1$ and generalizes the ordinary derivative $p'(\xi)$ as,

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(\xi)}{\alpha} = p'(\xi).$$

Aziz [2] extends inequality (9) to polar derivative version by obtaining the following result.

Theorem 1.3. *If $p(\xi) \in P_m$ non-vanishing in $|\xi| < r$, $r \ge 1$, then for any real or complex number α with $|\alpha| \ge 1$,*

$$\|D_\alpha p\|_\infty \le m\left(\frac{|\alpha|+r}{1+r}\right)\|p\|_\infty. \tag{22}$$

It is clearly of interest to prove the L^γ inequalities for polar derivative of the polynomial. Likewise, Govil et al. [13] generalize inequality (7) proved by de Brujin [7] and (22) due to Aziz [2] for $r = 1$ by proving,

Theorem 1.4. *If $p(\xi) \in P_m$ has no zero in $|\xi| < 1$, then for all real or complex number α with $|\alpha| \ge 1$ and for $\gamma \ge 1$,*

$$\|D_\alpha p\|_\gamma \le m\frac{(|\alpha|+1)}{\|1+\xi\|_\gamma}\|p\|_\gamma. \tag{23}$$

Further, Aziz et al. [5] extended (22) due to Aziz [2] to L^γ analogue for the polar derivative by obtaining,

Theorem 1.5. *If $p(\xi) \in P_m$ has no zero in $|\xi| < r$, $r \ge 1$, then for $\gamma \ge 1$ and for any real or complex number α with $|\alpha| \ge 1$,*

$$\|D_\alpha p\|_\gamma \le m\frac{(|\alpha|+r)}{\|r+\zeta\|_\gamma}\|p\|_\gamma. \tag{24}$$

On the other hand, Aziz and Rather [3] obtained an extension of (15) to polar derivative. In fact, he proved the following result.

Theorem 1.6. *If $p(\xi) \in P_m$ has all its zeros in $|\xi| \le r$, $r \le 1$, then for all real or complex number α with $|\alpha| \ge k$,*

$$\|D_\alpha p\|_\infty \ge m\left(\frac{|\alpha|-r}{1+r}\right)\|p\|_\infty. \tag{25}$$

Dewan et al. [8] recently proved the result below which gives the extension of inequality (25) to L^γ norm inequality.

Theorem 1.7. *If $p \in P_m$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, then for any real or complex number α with $|\alpha| \geq r$ and for every $\gamma > 0$,*

$$\|D_\alpha p\|_\infty \geq m \frac{(|\alpha| - r)}{\|1 + r\zeta\|_\gamma} \|p\|_\gamma. \tag{26}$$

More information on the literature can be availed on the recent book of Gardner et. al [9].

2 Main Results

In this paper, we shall obtain L^γ norm inequality for the polar derivative of polynomial $p \in P_m$. Our results give implications to several known inequalities along with the inequalities we have discussed above.

We first prove the following result which is the L^γ analogue for the polar derivative of Theorem 1.1. It not only generalizes Theorem 1.1 but also improves inequality (24) due to Aziz et al. [5] and a result due to Mir and Ahmad [22, Corollary 1] particularly for $l = 1$ directly, while its ordinary version improves inequalities (10) and (13).

Theorem 2.1. *If $p \in P_m$ has no zero in $|\xi| < r$, $r \geq 1$, then for any real or complex number α with $|\alpha| \geq 1$ and for every $\gamma > 0$,*

$$\|D_\alpha p\|_\gamma \leq m \frac{(C + |\alpha|)}{\|C + \xi\|_\gamma} \|p\|_\gamma, \tag{27}$$

where,

$$C = \frac{r(1 - |\lambda|)(|\lambda| + r^2) + r(n-1)|l - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda| r^2) + r(n-1)|l - \lambda^2|}, \tag{28}$$

$$\lambda = \frac{r}{m} \frac{c_1}{c_0}, \quad \textit{and} \quad l = \frac{2r^2}{m(m-1)} \frac{c_2}{c_0},$$

such that,

$$|\lambda| \leq 1, \quad \textit{and} \quad (m-1)|l - \lambda^2| \leq 1 - |\lambda|^2.$$

Remark 2.1. *It is of interest to verify the fact,*

$$C \geq r, \tag{29}$$

and

$$C \geq \delta_{r,1}, \tag{30}$$

where $r \geq 1$ and C and $\delta_{r,1}$ are as defined in (28) of Theorem 2.1 and inequality (14) respectively.

In order to verify $C \geq r$, it is sufficient to show that,

$$\frac{(1 - |\lambda|)(|\lambda| + r^2) + r(m-1)|l - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda| r^2) + r(m-1)|l - \lambda^2|} \geq 1,$$

$$i.e. \qquad |\lambda| + r^2 \geq 1 + |\lambda| r^2,$$

$$i.e. \qquad r^2(1 - |\lambda|) \geq 1 - |\lambda|,$$

$$i.e. \qquad r^2 \geq 1,$$

since $1 - |\lambda| \geq 0$ (by Lemma 3.1) and $r \geq 1$.

Further, we have

$$\delta_{r,1} = \frac{m|c_0|r^2 + |c_1|r^2}{m|c_0| + |c_1|r^2} = \frac{r + |\lambda|}{r + |\lambda|r^2},$$

where λ is as defined in Theorem 2.1.
To show that $C \geq \delta_{r,1}$, it is sufficient enough by showing that,

$$\frac{r(1 - |\lambda|)(|\lambda| + r^2) + r^2(m-1)|l - \lambda^2|}{(1 - |\lambda|)(1 + |\lambda|r^2) + r(m-1)|l - \lambda^2|} \geq \frac{r + |\lambda|}{r + |\lambda|r^2},$$

which implies,

$$(r-1)\Bigg[(1 - |\lambda|)\Big\{r^3|\lambda|(r+1) + r^2|\lambda|^2 + r^3 + |\lambda|(r+1)\Big\} + r(m-1)|l - \lambda^2|\Big\{r + |\lambda|(r^2 + r + 1)\Big\}\Bigg] \geq 0,$$

from which we eventually obtain,

$$(1 - |\lambda|)\Big\{r^4|\lambda| + r^3(|\lambda| + 1) + r^2|\lambda|^2 + r|\lambda| + |\lambda|\Big\} + r(m-1)|l - \lambda^2|\Big\{r^2|\lambda| + r(|\lambda| + 1) + |\lambda|\Big\} \geq 0,$$

which is true by the fact that $|\lambda| \leq 1$, since by Lemma 3.1.

Remark 2.2. *Inequality (27) of Theorem 2.1 is an improvement of inequality (24) due to Aziz et al. [5] and it follows on applying the fact of inequality (29) and Lemma 3.5 with $a = |\alpha| \geq 1$, $b = C$ and $c = r$,*

$$\frac{|\alpha| + C}{\left\{\int_0^{2\pi} |e^{i\theta} + C|^\gamma d\theta\right\}^{\frac{1}{\gamma}}} \leq \frac{|\alpha| + r}{\left\{\int_0^{2\pi} |e^{i\theta} + r|^\gamma d\theta\right\}^{\frac{1}{\gamma}}},$$

which clearly shows that inequality (27) sharpens (24).

Following similar argument as above and noting the fact of inequality (30), it is clear that inequality (27) of Theorem 2.1 further improves a result by Mir and Ahmad [22, Corollary 1] particularly for $l = 1$.

Remark 2.3. *When dividing by $|\alpha|$ on both sides of (27) and letting $|\alpha| \to \infty$, we obtain the following corollary which is an improvement of inequalities (10) and (13) recently proved by Krishnadas and Chanam [17, Theorem 1].*

Corollary 2.1. *If $p \in P_m$ has no zero in $|\xi| < r$, $r \geq 1$, then for every $\gamma > 0$,*

$$\|p'\|_\gamma \leq \frac{m}{\|C + \xi\|_\gamma}\|p\|_\gamma, \tag{31}$$

where C is as defined in Theorem 2.1.

Remark 2.4. *Further, if we let $\gamma \to \infty$ in inequality (31), we obtain,*

$$\|p'\|_\infty \leq \frac{m}{C + 1}\|p\|_\infty, \tag{32}$$

which is inequality (19) of Theorem 1.1 due to Govil et al. [15].

Remark 2.5. *It follows readily from the fact* (29) *and* (30) *of Remark* 2.1 *that inequality* (31) *of Corollary* 2.1 *improves both inequalities* (9) *and* (12) *respectively proved by Malik* [19] *and Govil et al.* [15].

Next, we consider the L^γ analogue for polar derivative of the polynomials $p \in P_m$ having all its zeros in $|\xi| \le r$, $r \le 1$ and we prove the following theorem which gives an improvement of inequality (26) due to Dewan et al. [8] and a result due to Mir [21, Corollary 1.2].

Theorem 2.2. *If $p \in P_m$ has all its zeros in $|\xi| \le r$, $r \le 1$, then for any real or complex number α with $|\alpha| \ge D$ and for every $\gamma > 0$,*

$$\|D_\alpha p\|_\gamma \ge m\frac{(|\alpha| - D)}{\|1 + D\xi\|_\gamma}\|p\|_\gamma, \tag{33}$$

where,

$$D = r\frac{(1-|\omega|)(|\omega| + r^2) + r(n-1)|\Omega - \omega^2|}{(1-|\omega|)(1+|\omega|r^2) + r(m-1)|\Omega - \omega^2|}, \tag{34}$$

$$\omega = \frac{1}{mr}\frac{\bar{c}_{m-1}}{\bar{c}_m}, \qquad and \qquad \Omega = \frac{2}{m(m-1)r^2}\frac{\bar{c}_{m-2}}{\bar{c}_m},$$

and

$$|\omega| \le 1, \qquad and \qquad (m-1)|\Omega - \omega^2| \le 1 - |\omega|^2.$$

Remark 2.6. *To prove that,*

$$D \le t_{r,1}, \tag{35}$$

where D and $t_{r,1}$ are as defined in (34) *in Theorem* 2.2 *and inequality* (18) *respectively. Since,*

$$t_{r,1} = \frac{m|c_m|r^2 + |c_{m-1}|}{m|c_m| + |c_{m-1}|} = \frac{r + |\omega|}{r + r^2|\omega|},$$

$D \le t_{r,1}$ implies

$$\frac{r(1-|\omega|)(|\omega| + r^2) + r^2(m-1)|\Omega - \omega^2|}{(1-|\omega|)(1+|\omega|r^2) + r(m-1)|\Omega - \omega^2|} \le \frac{r + |\omega|}{r + r^2|\omega|}.$$

Simplifying the above inequality, we get

$$(r-1)\Bigg[(1-|\omega|)\Big\{r^3|\omega|(r+1) + r^2|\omega|^2 + r(r^2+r+1) + |\omega|(r+1)\Big\} + r(m-1)|\Omega - \omega^2|\Big\{r + |\omega|(r^2+r+1)\Big\}\Bigg] \le 0.$$

Since $r \le 1$, we have

$$(1-|\omega|)\Big\{r^3|\omega|(r+1) + r^2|\omega|^2 + r(r^2+r+1) + |\omega|(r+1)\Big\} + r(m-1)|\Omega - \omega^2|\Big\{r + |\omega|(r^2+r+1)\Big\} \ge 0,$$

and is true as $|\omega| \le 1$ by Lemma 3.2.

Remark 2.7. *Inequality* (33) *of Theorem* 2.2 *is an improvement of* (26) *due to Dewan et al.* [8] *and it follows on applying the fact of inequality* (35) *and Lemma* 3.6 *with $a = |\alpha| \ge 1$, $b = D$ and $c = r$,*

$$\frac{|\alpha| - D}{\left\{\int_0^{2\pi}|1 + De^{i\theta}|^\gamma d\theta\right\}^{\frac{1}{\gamma}}} \ge \frac{|\alpha| - r}{\left\{\int_0^{2\pi}|1 + re^{i\theta}|^\gamma d\theta\right\}^{\frac{1}{\gamma}}},$$

which clearly shows that inequality (33) sharpens (26).

Following similar argument as above and noting the fact of inequality (35), it is clear that inequality (33) of Theorem 2.2 further improves a result due to Mir [21, Corollary 1.2].

Remark 2.8. *When dividing both the sides of (33) by $|\alpha|$ and making $|\alpha| \to \infty$, we obtain a result proved by Krishnadas and Chanam [17, Theorem 2] recently.*

Corollary 2.2. *If $p \in P_m$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, then for every $\gamma > 0$,*

$$\|p'\|_\gamma \geq \frac{m}{\|1 + D\xi\|_\gamma}\|p\|_\gamma, \tag{36}$$

where D is as defined in Theorem 2.2.

Since $|p'(e^{i\theta})| \leq \|p'\|_\infty$ for $0 \leq \theta < 2\pi$, we can immediately obtain the following corollary.

Corollary 2.3. *If $p \in P_m$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, then for every $\gamma > 0$,*

$$\|p'\|_\infty \geq \frac{m}{\|1 + D\xi\|_\gamma}\|p\|_\gamma, \tag{37}$$

where D is defined in Theorem 2.2.

Remark 2.9. *From the fact that $D \leq t_{r,1}$, Corollary 2.3 is an improvement of inequality (17) proved by Aziz and Rather [4, Corollary 4].*

Remark 2.10. *We know by definition,*

$$\frac{m}{\|1 + D\xi\|_\infty} = m \lim_{\gamma\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|1 + De^{i\theta}|^\gamma d\theta\right\}^{-\frac{1}{\gamma}} = \frac{m}{1 + D},$$

by (34), it further simplifies to

$$\frac{m}{1 + r}\frac{(1 - |\omega|)(1 + r^2|\omega|) + r(m - 1)|\Omega - \omega^2|}{(1 - |\omega|)(1 - r + r^2 + r|\omega|) + r(m - 1)|\Omega - \omega^2|}. \tag{38}$$

Thus, when letting $\gamma \to \infty$, (37) reduces to (20) of Theorem 1.2.

Remark 2.11. *The ordinary derivative inequality form (20) of inequality (37) obtained from taking the limit as $\gamma \to \infty$ in (37) is an improvement of inequality (15) due to Malik [19]. To show this, it is sufficient that,*

$$\frac{(1 - |\omega|)(1 + r^2|\omega|) + r(m - 1)|\Omega - \omega^2|}{(1 - |\omega|)(1 - r + r^2 + r|\omega|) + r(m - 1)|\Omega - \omega^2|} \geq 1,$$

which is equivalent to

$$(1 - |\omega|)(1 - r) \geq 0, \text{ which is true by hypotheses of Theorem 2.2.}$$

3 Lemmas

In proving our results, the following lemmas are needed.

Lemma 3.1. *If $p \in P_m$ has no zero in $|\xi| < r$, $r \geq 1$, then,*

$$C|p'(\xi)| \leq |y'(\xi)|, \tag{39}$$

where here and elsewhere $y(\xi) = \xi^m \overline{p(\frac{1}{\bar{\xi}})}$,

$$C = \frac{r(1-|\lambda|)(|\lambda|+r^2) + r(m-1)|l-\lambda^2|}{(1-|\lambda|)(1+|\lambda|r^2) + r(m-1)|l-\lambda^2|},$$

$$\lambda = \frac{r}{m}\frac{c_1}{c_0}, \quad \text{and} \quad l = \frac{2r^2}{m(m-1)}\frac{c_2}{c_0},$$

such that,

$$|\lambda| \leq 1, \quad \text{and} \quad (m-1)|l-\lambda^2| \leq 1-|\lambda|^2.$$

Govil et al. [15] obtained the above lemma.

Lemma 3.2. *If $p \in P_m$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, then on $|\xi| = 1$,*

$$|y'(\xi)| \leq D|p'(\xi)|, \tag{40}$$

where,

$$D = r\frac{(1-|\omega|)(|\omega|+r^2) + r(m-1)|\Omega-\omega^2|}{(1-|\omega|)(1+|\omega|r^2) + r(m-1)|\Omega-\omega^2|},$$

$$\omega = \frac{1}{mr}\frac{\bar{c}_{m-1}}{\bar{c}_m}, \quad \text{and} \quad \Omega = \frac{2}{m(m-1)r^2}\frac{\bar{c}_{m-2}}{\bar{c}_m},$$

and

$$|\omega| \leq 1, \quad \text{and} \quad (m-1)|\Omega-\omega^2| \leq 1-|\omega|^2.$$

This lemma is due to Krishnadas and Chanam [17].

Lemma 3.3. *Let ξ_1 and ξ_2 be any two complex numbers not depending on β, β is real. Then, for each $\gamma > 0$,*

$$\int_0^{2\pi} \left|\xi_1 + \xi_2 e^{i\beta}\right|^\gamma d\beta = \int_0^{2\pi} \left||\xi_1| + |\xi_2|e^{i\beta}\right|^\gamma d\beta. \tag{41}$$

This lemma is due to Gardner and Govil [10].

Govil and Kumar [12] obtains the following two lemmas.

Lemma 3.4. *Let p, q be any two positive numbers such that $p \geq qx$, where $x \geq 1$. If β is any real number in $[0, 2\pi]$, then,*

$$\frac{p+qy}{x+y} \leq \left|\frac{p+qe^{i\beta}}{x+e^{i\beta}}\right|, \tag{42}$$

for $y \geq 1$.

Lemma 3.5. *If $a \geq 1$, $b \geq c \geq 1$ and $\gamma > 0$, then,*

$$\frac{a+b}{\left\{\int_0^{2\pi} |e^{i\theta}+b|^\gamma d\theta\right\}^{\frac{1}{\gamma}}} \leq \frac{a+c}{\left\{\int_0^{2\pi} |e^{i\theta}+c|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}. \tag{43}$$

Lemma 3.6. *If $a \geq c$, $b \leq c \leq 1$ and $\gamma > 0$, then,*

$$\frac{a-b}{\left\{\int_0^{2\pi} |1+be^{i\theta}|^\gamma d\theta\right\}^{\frac{1}{\gamma}}} \geq \frac{a-c}{\left\{\int_0^{2\pi} |1+ce^{i\theta}|^\gamma d\theta\right\}^{\frac{1}{\gamma}}}. \tag{44}$$

Proof. When $a = c$, inequality (44) follows trivially.

Suppose $a > c$, then it suffices to show that,

$$\int_0^{2\pi} \left(\frac{|1+be^{i\theta}|}{a-b}\right)^\gamma d\theta \leq \int_0^{2\pi} \left(\frac{|1+ce^{i\theta}|}{a-c}\right)^\gamma d\theta,$$

for which we will show,

$$\left(\frac{|1+be^{i\theta}|}{a-b}\right)^\gamma \leq \left(\frac{|1+ce^{i\theta}|}{a-c}\right)^\gamma, \tag{45}$$

for all $\theta \in [0, 2\pi]$ and $a > x$, $b \leq c \leq 1$.

To prove (45), we take the function

$$f : x \to \frac{|1+xe^{i\theta}|}{a-x},\ a > x,$$

on $[0, 1]$, and prove that f is non-decreasing. We can see that,

$$f'(x) \geq 0 \ \text{ if and only if } \ ax + 1 + (a+x)\cos\theta \geq 0,$$

which is true, because if $ax + 1 + (a+x)\cos\theta < 0$, and with simple rearrangement, the above implies,

$$a < -\frac{1 + x\cos\theta}{x + \cos\theta}. \tag{46}$$

Since $|1 + xe^{i\theta}| \geq 0, \forall\, x \in R$, then the r.h.s of (46) is equal to or less than x, implying that $a < x$, which contradicts to the fact that $a \geq x$. Thus for all real values of θ, we have $ax+1+(a+x)\cos\theta \geq 0$, implies that f and f^γ are increasing, from which inequality (44) follows. □

Lemma 3.7. *Let $p(\xi) \in P_m$ and $y(\xi) = \xi^m \overline{p(\frac{1}{\bar{\xi}})}$, then for every β, $0 \leq \beta < 2\pi$ and $\gamma > 0$,*

$$\int_0^{2\pi} \int_0^{2\pi} |p'(e^{i\theta}) + e^{i\beta} y'(e^{i\theta})|^\gamma d\theta d\beta \leq 2\pi m^\gamma \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta. \tag{47}$$

This lemma was obtained by Aziz and Rather [4].

4 Proof of Theorems

Proof of Theorem 2.1:
For $\gamma > 0$, we have

$$\begin{aligned}\left(\int_0^{2\pi}\left|D_\alpha p(e^{i\theta})\right|^\gamma d\theta\right)\left(\int_0^{2\pi}\left|C+e^{i\beta}\right|^\gamma d\beta\right)\\ &= \int_0^{2\pi}\int_0^{2\pi}\left|C+e^{i\beta}\right|^\gamma\left|D_\alpha p(e^{i\theta})\right|^\gamma d\theta d\beta,\\ &= \int_0^{2\pi}\int_0^{2\pi}\left|C+e^{i\beta}\right|^\gamma\left|mp(e^{i\theta})-e^{i\theta}p'(e^{i\theta})+\alpha p'(e^{i\theta})\right|^\gamma d\theta d\beta,\\ &\le \int_0^{2\pi}\int_0^{2\pi}\left|C+e^{i\beta}\right|^\gamma\left\{\left|mp(e^{i\theta})-e^{i\theta}p'(e^{i\theta})\right|+|\alpha||p'(e^{i\theta})|\right\}^\gamma d\theta d\beta,\\ &\le \int_0^{2\pi}\int_0^{2\pi}\left|C+e^{i\beta}\right|^\gamma\left\{|y'(e^{i\theta})|+|\alpha||p'(e^{i\theta})|\right\}^\gamma d\theta d\beta. \end{aligned} \tag{48}$$

Now by Lemma 3.1, we have

$$C|p'(e^{i\theta})| \le |y'(e^{i\theta})|. \tag{49}$$

Further, we can verify that,

$$\left||y'(e^{i\theta})|+e^{i\beta}|p'(e^{i\theta})|\right| = \left||p'(e^{i\theta})|+e^{i\beta}|y'(e^{i\theta})|\right|. \tag{50}$$

Comparing with Lemma 3.3, we have

$$p \equiv |y'(e^{i\theta})|, \qquad q \equiv |p'(e^{i\theta})| \qquad \text{and} \qquad x \equiv C.$$

Also by assumption, $|\alpha| > 1$. The role of y is taken by $|\alpha|$ i.e., $y = |\alpha|$. By Lemma 3.4, we have

$$\frac{p+qy}{x+y} \le \left|\frac{p+qe^{i\beta}}{x+e^{i\beta}}\right|, \qquad \forall \quad \beta \in [0, 2\pi],$$

which implies,

$$\frac{|y'(e^{i\theta})|+|\alpha||p'(e^{i\theta})|}{(C+|\alpha|)} \le \left|\frac{|y'(e^{i\theta})|+|p'(e^{i\theta})|e^{i\beta}}{(C+e^{i\beta})}\right|. \tag{51}$$

By cross multiplication, we get

$$|C+e^{i\beta}|\left\{|y'(e^{i\theta})|+|\alpha||p'(e^{i\theta})|\right\} \le (C+|\alpha|)\left\{\left||y'(e^{i\theta})|+e^{i\beta}|p'(e^{i\theta})|\right|\right\},$$

and using inequality (50), we have

$$|C+e^{i\beta}|\left\{|y'(e^{i\theta})|+|\alpha||p'(e^{i\theta})|\right\} \le (C+|\alpha|)\left\{\left||p'(e^{i\theta})|+e^{i\beta}|y'(e^{i\theta})|\right|\right\}. \tag{52}$$

Using inequality (52) in (48) and by Lemma 3.3, we get

$$\begin{aligned}\int_0^{2\pi}|D_\alpha p(e^{i\theta})|^\gamma d\theta\int_0^{2\pi}|C+e^{i\beta}|^\gamma d\beta\\ &\le (C+|\alpha|)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left||p'(e^{i\theta})|+e^{i\beta}|y'(e^{i\theta})|\right|^\gamma d\theta d\beta,\\ &= (C+|\alpha|)^\gamma\int_0^{2\pi}\int_0^{2\pi}\left|p'(e^{i\theta})+e^{i\beta}y'(e^{i\theta})\right|^\gamma d\theta d\beta. \end{aligned} \tag{53}$$

Applying Lemma 3.7 to (53) gives,

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\gamma d\theta\right)\left(\int_0^{2\pi} |C+e^{i\beta}|^\gamma d\beta\right) \leq (C+|\alpha|)^\gamma 2\pi m^\gamma \left(\int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta\right),$$

which is equivalent to

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\gamma d\theta\right)^{\frac{1}{\gamma}} \leq m\frac{(C+|\alpha|)}{\left(\frac{1}{2\pi}\int_0^{2\pi} |C+e^{i\beta}|^\gamma d\beta\right)^{\frac{1}{\gamma}}}\left(\int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta\right)^{\frac{1}{\gamma}}, \tag{54}$$

which completes the proof of Theorem 2.1.

Proof of Theorem 2.2:
Since $p(\xi)$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, $p'(\xi)$ has all its zeros in $|\xi| \leq r$, $r \leq 1$. Hence, by Gauss-Lucas Theorem, the polynomial

$$\xi^{m-1}\overline{p'\left(\frac{1}{\bar{\xi}}\right)} = my(\xi) - \xi y'(\xi), \tag{55}$$

has all its zeros in $|\xi| \geq \frac{1}{r}, \frac{1}{r} \geq 1$.

Further, since $p(\xi)$ has all its zeros in $|\xi| \leq r$, $r \leq 1$, by Lemma 3.2, we have

$$|y'(\xi)| \leq D|p'(\xi)| \text{ for } |\xi| = 1, \tag{56}$$

where D is as defined by (34).

For $|\xi| = 1$, we also have

$$|p'(\xi)| = \Big|my(\xi) - \xi y'(\xi)\Big|. \tag{57}$$

Using (56) in (57), we have on $|\xi| = 1$,

$$|y'(\xi)| \leq D\Big|my(\xi) - \xi y'(\xi)\Big|. \tag{58}$$

Therefore, it follows from (56) that the function

$$\phi(\xi) = \frac{\xi y'(\xi)}{D\left\{my(\xi) - \xi y'(\xi)\right\}}, \tag{59}$$

is analytic in $|\xi| \leq 1$, $|\phi(\xi)| \leq 1$ on $|\xi| = 1$ and $\phi(0) = 0$. Thus, the function $1+D\phi(\xi)$ is subordinate to the function $1 + D\xi$ for $|\xi| \leq 1$. Hence, from a well-known property of subordination [16], we have for every $\gamma > 0$,

$$\int_0^{2\pi} \left|1 + D\phi(e^{i\theta})\right|^\gamma d\theta \leq \int_0^{2\pi} \left|1 + De^{i\theta}\right|^\gamma d\theta. \tag{60}$$

Now,

$$1 + D\phi(\xi) = 1 + \frac{\xi y'(\xi)}{my(\xi) - \xi y'(\xi)} = \frac{\xi y(\xi)}{my(\xi) - \xi y'(\xi)}, \tag{61}$$

which implies for $|\xi| = 1$,

$$\begin{aligned} |my(\xi)| &= \left|1 + D\phi(\xi)\right|\left|my(\xi) - \xi y'(\xi)\right|, \\ &= \left|1 + D\phi(\xi)\right|\left|p'(\xi)\right|. \quad \text{(by (57))} \end{aligned} \tag{62}$$

Since $|p(\xi)| = |y(\xi)|$ on $|\xi| = 1$, we have from the proceeding inequality

$$m|p(\xi)| = \big|1 + D\phi(\xi)\big||p'(\xi)| \text{ on } |\xi| = 1. \tag{63}$$

Also, by the definition of polar derivative of a polynomial, we have

$$D_\alpha p(\xi) = mp(\xi) + (\alpha - \xi)p'(\xi),$$

from which we have for $|\xi| = 1$,

$$\begin{aligned} |D_\alpha p(\xi)| &\geq \Big||\alpha||p'(\xi)| - |mp(\xi) - \xi p'(\xi)|\Big|, \\ &= \Big||\alpha||p'(\xi)| - |y'(\xi)|\Big|, \quad \Big[\text{since for } |\xi| = 1, |y'(\xi)| = |mp(\xi) - \xi p'(\xi)|\Big] \end{aligned} \tag{64}$$

By using (56) on r.h.s of (64), we get

$$|\alpha||p'(\xi)| - |y'(\xi)| \geq \big(|\alpha| - D\big)|p'(\xi)|,$$

which is non-negative, since $|\alpha| \geq D$. Thus, (64) gives

$$|D_\alpha p(\xi)| \geq \big(|\alpha| - D\big)|p'(\xi)|. \tag{65}$$

Using (65) in (63), we get

$$m|p(\xi)| \leq \big|1 + D\phi(\xi)\big|\frac{|D_\alpha p(\xi)|}{\big(|\alpha| - D\big)}, \qquad \text{for } |\xi| = 1.$$

From which equivalently we conclude that for each θ, $0 \leq \theta < 2\pi$, and for each $\gamma > 0$,

$$m^\gamma(|\alpha| - D)^\gamma \int_0^{2\pi} \frac{|p(e^{i\theta})|^\gamma}{|D_\alpha p(e^{i\theta})|^\gamma} d\theta \leq \int_0^{2\pi} |1 + D\phi(e^{i\theta})|^\gamma d\theta, \tag{66}$$

which on using (60) gives

$$m(|\alpha| - D)\left(\int_0^{2\pi} \frac{|p(e^{i\theta})|^\gamma}{|D_\alpha p(e^{i\theta})|^\gamma} d\theta\right)^{\frac{1}{\gamma}} \leq \left(\int_0^{2\pi} |1 + De^{i\theta}|^\gamma d\theta\right)^{\frac{1}{\gamma}},$$

which is equivalent to

$$\left(\int_0^{2\pi} |D_\alpha p(e^{i\theta})|^\gamma d\theta\right)^{\frac{1}{\gamma}} \geq m\frac{(|\alpha| - D)}{\left(\frac{1}{2\pi}\int_0^{2\pi} |1 + De^{i\theta}|^\gamma d\theta\right)^{\frac{1}{\gamma}}}\left(\int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta\right)^{\frac{1}{\gamma}}, \tag{67}$$

which completes the proof of Theorem 2.2.

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